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Mixed-mode stress intensity factors for a crack in an anisotropic bi-material strip

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Abstract

This paper provides a method for obtaining the mixed-mode stress intensity factors for a bi-material interface crack in the infinite strip configuration and in the case where both phases are fully anisotropic. First, the dislocation solution in a bi-material anisotropic infinite strip is investigated (the boundary of the strip is parallel to the bi-material interface). A surface distributed dislocation approach is employed to ensure the traction-free conditions at the strip bounding surfaces. Subsequently, the derived dislocation solution is applied to calculate the mixed-mode stress intensity factors of a crack located at, or parallel to, the interface in the bi-material anisotropic infinite strip. The crack itself is modelled as a distribution of the derived dislocation solutions for the strip. Results are presented and the effects of material mismatch, the length of the crack and the material interface on the stress intensity factors are investigated.

Keywords: Anisotropic; Crack; Fracture; Composites; Bi-material; Strip; Stress Intensity Factors

1. Introduction

Anisotropic bi-materials are often encountered in modern technology with the increasing use of composite and sandwich material systems. The fracture behavior at the interface between these dissimilar materials (namely the different layers of the composite) is a critical phenomenon and frequently the weak link in the safe and confident use of these modern materials. Determining the stress intensity factors of interface cracks in anisotropic bi-materials is the first step in predicting the subsequent crack propagation and damage tolerance. One important point is that the construction with these composite and sandwich systems typically involves the configuration of more or less thin "strip" geometry, therefore the commonly encountered in the literature formulations and results on infinite plane or half-plane configurations would not normally be applicable.

One of the most effective methods in anisotropic fracture mechanics is the distributed dislocation technique, which is a semi-analytical technique and has been already effectively used by Huang and Kardomateas

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(1999, 2001). The basic idea of this method is to model the crack as an array of dislocations along the crack line and determine the dislocation densities by satisfying the crack surface traction free conditions. The mixed-mode stress intensity factors can subsequently be calculated from the dislocation densities.

The cornerstone of this method is the fundamental solution of a dislocation in the corresponding configuration. Eshelby et al. (1953) and Stroh (1958) are among the pioneers who presented analytical solutions for a dislocation in general anisotropic materials. Following their work, Ting (1986), and Qu and Li (1991) studied the classical problem of a dislocation situated at the interface between two anisotropic elastic half planes and obtained an analytical solution to the dislocation problem. Atkinson and Eftaxiopoulos (1991) also achieved the solution for a dislocation in an anisotropic half plane and a bi-material infinite plane, using the basic formulation of Stroh. Bi-material half-planes with a crack located at or parallel to the interface have been studied by Huang and Kardomateas (2001) by use of the distributed dislocation technique.

As far as the strip geometry, Civelek and Erdogan (1982) developed a numerical method to calculate the dislocation solution in an *isotropic homogeneous* infinite strip by superposing the infinite plane with an additional elastic field, which is expressed by an Airy stress function with Fourier transformation. Suo (1990) and Suo and Hutchinson (1990) extended this method to *orthotropic* materials and calculated the mixed-mode stress intensity factors for an infinite strip with semi-infinite cracks subjected to edge bending. Their technique, undoubtedly quite elegant, is limited to orthotropic materials. Huang and Kardomateas (1999) developed a method to calculate the stress fields of a dislocation in a *homogeneous* anisotropic infinite strip and applied the solution to calculate the stress intensity factors for both single edge and double edge cracks in a fully anisotropic homogeneous infinite strip. But analytical studies of the *fully anisotropic bi-material strip* cannot be found in the literature, short of finite element results associated with interlaminar cracks in composite laminates (Qian and Sun, 1997).

In this paper, first the analytical solution for a dislocation in an anisotropic bi-material infinite plane is summarized and then the stress field for a dislocation in an anisotropic bi-material strip is obtained by distributing two dislocation arrays along the traction-free boundaries of the infinite strip. The dislocation solution for the strip thus derived, is then applied to calculate the mixed mode stress intensity factors for a crack located at, or parallel to, the interface of the bi-material anisotropic infinite strip. This last step involves modelling the crack itself as a distribution of the derived dislocation solutions for the strip.

2. Formulation

2.1. Dislocation solution in a bi-material anisotropic infinite plane

The analytical solution for dislocations in a bi-material infinite plane has several different versions. Almost all of them originate from Stroh's formulation. Combining the solutions presented by Ting (1986), Qu and Li (1991), and Atkinson and Eftaxiopoulos (1991), we present first a concise summary of this elegant analytical solution for a dislocation in a bi-material infinite plane.

In a homogenous anisotropic medium, the constitutives are:

$$\sigma_{ij} = C_{ijkl}\frac{\partial U_k}{\partial x_l}, \tag{1}$$

where $i, j, k, l = 1, 2, 3$ and the Einstein indices convention applies. C_{ijkl} is the elastic stiffness tensor and it satisfies: $C_{ijkl} = C_{jikl}$; $C_{ijkl} = C_{ijlk}$.

The equilibrium equations can be written as

$$\frac{\partial \sigma_{ij}}{\partial x_j} = C_{ijkl}\frac{\partial^2 U_k}{\partial x_l \partial x_j} = 0, \tag{2}$$

which are the partial differential equations governing the stress field of the homogeneous anisotropic medium.

We can assume the displacement field in the following form, which satisfies Eq. (2):

$$U_k = A_k f(x_1 + Px_2), \tag{3}$$

provided that the constant A_k satisfies the equations:

$$(C_{i1k1} + PC_{i1k2} + PC_{i2k1} + P^2 C_{i2k2})A_k = 0. \tag{4}$$

$A_k \neq 0$ can be found if P is a root of the sextic equation (the determinant of the coefficients of (4)):

$$\mid C_{i1k1} + PC_{i1k2} + PC_{i2k1} + P^2 C_{i2k2} \mid = 0. \tag{5}$$

Then the displacements can be written as

$$U_k = \sum_\alpha A_{k\alpha} f_\alpha(z_\alpha) + \sum_\alpha \bar{A}_{k\alpha} \bar{f}_\alpha(\bar{z}_\alpha), \tag{6}$$

where

$$z_\alpha = x_1 + P_\alpha x_2 \quad \alpha = 1, 2, 3. \tag{7}$$

If ϕ_i is a function of x_1 and x_2 and the stresses:

$$\sigma_{i1} = -\frac{\partial \phi_i}{\partial x_2}; \quad \sigma_{i2} = \frac{\partial \phi_i}{\partial x_1}, \tag{8}$$

then, because the stresses should satisfy the equilibrium Eq. (3), from Eqs. (1), (7) and (8), we obtain:

$$\phi_i = \sum_\alpha L_{i\alpha} f_\alpha(z_\alpha) + \sum_\alpha \bar{L}_{i\alpha} \bar{f}_\alpha(\bar{z}_\alpha). \tag{9}$$

Then the stress components can be expressed as

$$\sigma_{i1} = -\sum_\alpha L_{i\alpha} P_\alpha f'_\alpha(z_\alpha) - \sum_\alpha \bar{L}_{i\alpha} \bar{P}_\alpha \bar{f}'_\alpha(\bar{z}_\alpha), \tag{10}$$

$$\sigma_{i2} = \sum_\alpha L_{i\alpha} f'_\alpha(z_\alpha) + \sum_\alpha \bar{L}_{i\alpha} \bar{f}'_\alpha(\bar{z}_\alpha), \tag{11}$$

where $L_{i\alpha}$ is defined by

$$L_{i\alpha} = (C_{i2k1} + P_\alpha C_{i2k2}) A_{k\alpha}. \tag{12}$$

All of the above relations are for an anisotropic medium. If there is a single dislocation $b = \{b_1, b_2, b_3\}$ located at $z_0(x_{10}, x_{20})$ in the medium, as we require that the stress components at the point of the dislocation be singular, we can choose a function:

$$f_\alpha(z_\alpha) = \frac{1}{4\pi} M_{\alpha j} d_j \ln(z_\alpha - z_{0\alpha}), \tag{13}$$

where

$$z_{0\alpha} = x_{10} + P_\alpha x_{20}, \tag{14}$$

$$b_i = B_{ij} d_j, \tag{15}$$

$$B_{ij} = \frac{1}{2} i \sum_\alpha (A_{i\alpha} M_{\alpha j} - \bar{A}_{i\alpha} \bar{M}_{\alpha j}), \tag{16}$$

$$M_{\alpha j} L_{j\beta} = \delta_{\alpha\beta}. \tag{17}$$

Then the displacements are given by

$$U_k = \sum_\alpha \frac{A_{k\alpha}}{4\pi} M_{\alpha j} d_j \ln(z_\alpha - z_{0\alpha}) + \sum_\alpha \frac{\bar{A}_{k\alpha}}{4\pi} \bar{M}_{\alpha j} d_j \ln(\bar{z}_\alpha - \bar{z}_{0\alpha}), \tag{18}$$

and the stresses are given by

$$\sigma_{i1} = -\frac{1}{4\pi}\left\{ \sum_\alpha L_{i\alpha} P_\alpha M_{\alpha j} \frac{d_j}{z_\alpha - z_{0\alpha}} + \sum_\alpha \bar{L}_{i\alpha} \bar{P}_\alpha \bar{M}_{\alpha j} \frac{d_j}{\bar{z}_\alpha - \bar{z}_{0\alpha}} \right\}, \tag{19}$$

$$\sigma_{i2} = \frac{1}{4\pi}\left\{ \sum_\alpha L_{i\alpha} M_{\alpha j} \frac{d_j}{z_\alpha - z_{0\alpha}} + \sum_\alpha \bar{L}_{i\alpha} \bar{M}_{\alpha j} \frac{d_j}{\bar{z}_\alpha - \bar{z}_{0\alpha}} \right\}. \tag{20}$$

For the anisotropic bi-material infinite plane, as shown in Fig. 1, there is a single dislocation $b = \{b_1, b_2, b_3\}^{\mathrm{T}}$ located in one of the homogeneous anisotropic media; here we assume it is located in medium (1). On the interface of the bi-materials, the tractions and displacements should be continuous i.e.:

$$U_k^{(1)}(x_1) = U_k^{(2)}(x_1), \tag{21}$$

$$\sigma_{i2}^{(1)}(x_1) = \sigma_{i2}^{(2)}(x_1). \tag{22}$$

Since for the homogeneous anisotropic medium (1) we require $f_\alpha^{(1)}(z_\alpha^{(1)})$ to be singular at $z_\alpha^{(1)} = z_{0\alpha}^{(1)}$ when $x_2 > 0$, and for the medium (2), we require $f_\alpha^{(2)}(z_\alpha^{(2)})$ to have no singularity when $x_2 < 0$, we can choose:

$$f_\alpha^{(1)}\left(z_\alpha^{(1)}\right) = \frac{1}{4\pi} M_{\alpha j}^{(1)} d_j^{(1)} \ln\left(z_\alpha^{(1)} - z_{0\alpha}\right) + \frac{1}{4\pi} \sum_\beta E_{\beta\alpha} \ln\left(z_\alpha^{(1)} - \bar{z}_{0\beta}^{(1)}\right), \tag{23}$$

$$f_\alpha^{(2)}\left(z_\alpha^{(2)}\right) = \frac{1}{4\pi} \sum_\beta G_{\beta\alpha} \ln\left(z_\alpha^{(2)} - z_{0\beta}^{(1)}\right), \tag{24}$$

where $E_{\beta\alpha}$ and $G_{\beta\alpha}$ are constants depending on the elastic properties of medium (1) and medium (2). Specifically, from Eqs. (21) and (22), we obtain:

$$A_{k\beta}^{(1)} M_{\beta j}^{(1)} d_j^{(1)} + \sum_\alpha \bar{A}_{k\alpha}^{(1)} \bar{E}_{\beta\alpha} = \sum_\alpha A_{k\alpha}^{(2)} G_{\beta\alpha}, \tag{25}$$

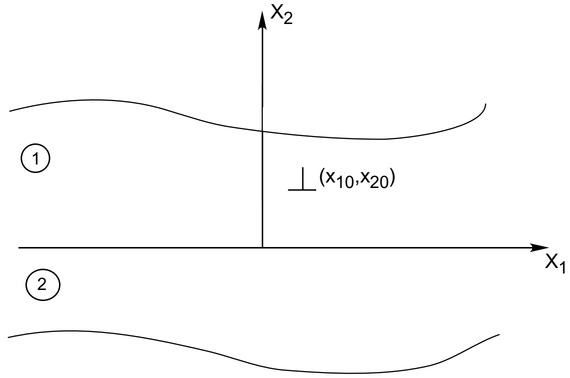

Fig. 1. Dislocation in a bi-material infinite plane.

$$L_{i\beta}^{(1)}M_{\beta j}^{(1)}d_j^{(1)} + \sum_\alpha \bar{L}_{i\alpha}^{(1)}\bar{E}_{\beta\alpha} = \sum_\alpha L_{i\alpha}^{(2)}G_{\beta\alpha}. \tag{26}$$

From Eqs. (25) and (26) we can determine $E_{\beta\alpha}$ and $G_{\beta\alpha}$, then the displacements and stresses are determined.

The stress components in medium (1) are written as

$$\sigma_{i1} = -\frac{1}{4\pi}\left\{\sum_\alpha L_{i\alpha}^{(1)}P_\alpha^{(1)}\left[M_{\alpha j}^{(1)}d_j^{(1)}\left(z_\alpha^{(1)} - z_{0\alpha}^{(1)}\right)^{-1} + \sum_\beta E_{\beta\alpha}\left(z_\alpha^{(1)} - \bar{z}_{0\beta}^{(1)}\right)^{-1}\right]\right\} + C.C. \tag{27}$$

$$\sigma_{i2} = \frac{1}{4\pi}\left\{\sum_\alpha L_{i\alpha}^{(1)}\left[M_{\alpha j}^{(1)}d_j^{(1)}\left(z_\alpha^{(1)} - z_{0\alpha}^{(1)}\right)^{-1} + \sum_\beta E_{\beta\alpha}\left(z_\alpha^{(1)} - \bar{z}_{0\beta}^{(1)}\right)^{-1}\right]\right\} + C.C. \tag{28}$$

where $C.C.$ means complex conjugate.

We assume that the medium (1) and (2) are linear elastic anisotropic materials, therefore we can express the stress components at $z = x_1 + ix_2$ due to a single dislocation $b = \{b_1, b_2, b_3\}^{\mathrm{T}}$ at $z_0(x_{10}, x_{20})$ as

$$\sigma_{ij}(x_1, x_2, x_{10}, x_{20}) = \mathbf{F_{ij}}(x_1, x_2, x_{10}, x_{20})\mathbf{b}(x_{10}, x_{20}), \tag{29}$$

where

$$\mathbf{F_{ij}}(x_1, x_2, x_{10}, x_{20}) = [f_{1ij}(x_1, x_2, x_{10}, x_{20}), f_{2ij}(x_1, x_2, x_{10}, x_{20}), f_{3ij}(x_1, x_2, x_{10}, x_{20})], \tag{30}$$

$$\mathbf{b}(x_{10}, x_{20}) = \{b_1(x_{10}, x_{20}), b_2(x_{10}, x_{20}), b_3(x_{10}, x_{20})\}^{\mathrm{T}}. \tag{31}$$

The physical meaning of $f_{lij}(x_1, x_2, x_{10}, x_{20})$ is that they are the stress components σ_{ij} due to a unit dislocation $b_l(x_{10}, x_{20})$; therefore we can calculate $f_{lij}(x_1, x_2, x_{10}, x_{20})$ from Eqs. (27) and (28) by setting $b_l(x_{10}, x_{20}) = \{1, 0, 0\}^{\mathrm{T}}$, $\{0, 1, 0\}^{\mathrm{T}}$, $\{0, 0, 1\}^{\mathrm{T}}$, respectively.

2.2. Dislocation solution in a bi-material anisotropic infinite strip

The bi-material infinite strip configuration shown in Fig. 2 consists of two anisotropic homogenous infinite strips of thickness h and H, respectively. The free boundaries are parallel to the material interface.

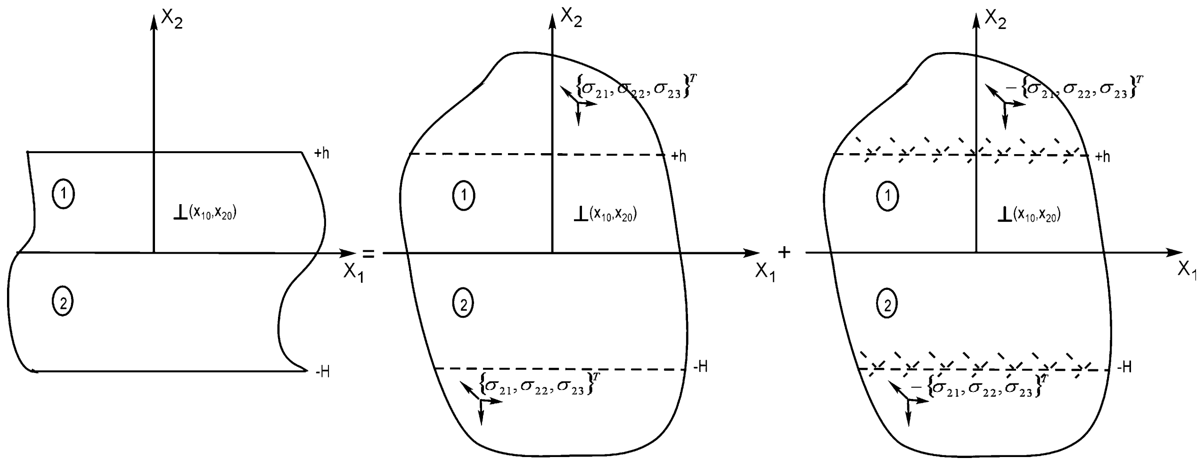

Fig. 2. An infinite strip as a distribution of dislocation arrays at the free boundaries.

The basic idea of obtaining the dislocation solution for the infinite strip is to apply two dislocation arrays along the free boundaries of the strip respectively. The densities of the dislocation arrays are determined in such a way that the tractions along the boundary due to a single dislocation and the dislocation arrays cancel out, therefore the boundaries are traction free.

The dislocation $b_0 = \{b_1, b_2, b_3\}^{\mathrm{T}}$ is located at an arbitrary point $z = x_{10} + \mathrm{i}x_{20}$. The geometry of a dislocation in the infinite strip can be decomposed into two configurations; the first one is a single dislocation located in the bi-material infinite plane. The dashed lines stand for the free boundaries of the infinite strip, which is supposed to be traction free. Then the traction forces along the dashed line due to the single dislocation $b_0(x_{10}, x_{20})$ can be determined from Eq. (29):

$$\sigma_{ij}^{(s)}\left(x_1^{(1)}, h, x_{10}, x_{20}\right) = \mathbf{F}_{ij}\left(x_1^{(1)}, h, x_{10}, x_{20}\right)\mathbf{b}_0(x_{10}, x_{20}), \tag{32}$$

$$\sigma_{ij}^{(s)}\left(x_1^{(2)}, -H, x_{10}, x_{20}\right) = \mathbf{F}_{ij}\left(x_1^{(2)}, -H, x_{10}, x_{20}\right)\mathbf{b}_0(x_{10}, x_{20}), \tag{33}$$

where $ij = 21, 22, 23$.

The second geometry is also the infinite plane with two dislocation arrays located along the supposed-to-be free boundaries of the infinite strip. In order to satisfy the traction free condition along the boundaries of the infinite strip, the tractions along the dashed lines in the second geometry should cancel out due to the single dislocation and the two dislocation arrays.

Assume the two dislocation arrays are distributed from $+\infty$ to $-\infty$, and then the stress components along the dashed line $x_2 = +h$ due to the two dislocation arrays can be calculated as

$$\sigma_{ij}^{(\mathrm{array})}\left(x_1^{(1)}, +h\right) = \int_{-\infty}^{+\infty} \mathbf{F}_{ij}\left(x_1^{(1)}, +h, s, +h\right)\mathbf{b}_{+h}(s, +h)\,\mathrm{d}s + \int_{-\infty}^{+\infty} \mathbf{F}_{ij}\left(x_1^{(1)}, +h, s, -H\right)\mathbf{b}_{-H}(s, -H)\,\mathrm{d}s. \tag{34}$$

These should cancel out by the traction forces due to the single dislocation $b_0(x_{10}, x_{20})$, therefore:

$$\sigma_{ij}^{(\mathrm{array})}\left(x_1^{(1)}, +h\right) = -\sigma_{ij}^{(s)}\left(x_1^{(1)}, +h\right). \tag{35}$$

Similarly

$$\sigma_{ij}^{(\mathrm{array})}\left(x_1^{(2)}, -H\right) = \int_{-\infty}^{+\infty} \mathbf{F}_{ij}\left(x_1^{(2)}, -H, s, +h\right)\mathbf{b}_{+h}(s, +h)\,\mathrm{d}s + \int_{-\infty}^{+\infty} \mathbf{F}_{ij}\left(x_1^{(2)}, -H, s, -H\right)\mathbf{b}_{-H}(s, -H)\,\mathrm{d}s. \tag{36}$$

Eqs. (34) and (36) are sets of singular integral equations. Gaussian quadrature is adopted to solve these equations numerically, and then the singular integral equations can be reduced to a set of algebraic equations. More detail derivation of the numerical solution of singular integral equations can be found in Hills et al. (1996). Eqs. (34) and (36) can be united and written in a matrix form:

$$\pi \begin{bmatrix} \bar{F}_{ij}\left(x_{1,k}^{(1)}, +h, s_m, +h\right) & \bar{F}_{ij}\left(x_{1,k}^{(1)}, +h, s_m, -H\right) \\ \bar{F}_{ij}\left(x_{1,k}^{(2)}, -H, s_m, +h\right) & \bar{F}_{ij}\left(x_{1,k}^{(2)}, -H, s_m, -H\right) \end{bmatrix} \langle \bar{W}_m \rangle \begin{bmatrix} \bar{b}_{+h}(\bar{s}_m, +h) \\ \bar{b}_{-H}(\bar{s}_m, -H) \end{bmatrix} = \begin{bmatrix} -\sigma_{21}^{(s)}\left(x_{1,k}^{(1)}, +h\right) \\ -\sigma_{22}^{(s)}\left(x_{1,k}^{(1)}, +h\right) \\ -\sigma_{23}^{(s)}\left(x_{1,k}^{(1)}, +h\right) \\ -\sigma_{21}^{(s)}\left(x_{1,k}^{(1)}, -H\right) \\ -\sigma_{22}^{(s)}\left(x_{1,k}^{(1)}, -H\right) \\ -\sigma_{23}^{(s)}\left(x_{1,k}^{(1)}, -H\right) \end{bmatrix}, \tag{37}$$

$$\pi\begin{bmatrix} \bar{F}_{ij}\left(x_{1,k}^{(1)},+h,s_m,+h\right) & \bar{F}_{ij}\left(x_{1,k}^{(1)},+h,s_m,-H\right) \\ \bar{F}_{ij}\left(x_{1,k}^{(2)},-H,s_m,+h\right) & \bar{F}_{ij}\left(x_{1,k}^{(2)},-H,s_m,-H\right) \end{bmatrix}$$

$$=\begin{bmatrix} \bar{F}_{21}\left(x_{1,k}^{(1)},+h,s_m,+h\right) & \bar{F}_{21}\left(x_{1,k}^{(1)},+h,s_m,-H\right) \\ \bar{F}_{22}\left(x_{1,k}^{(1)},+h,s_m,+h\right) & \bar{F}_{22}\left(x_{1,k}^{(1)},+h,s_m,-H\right) \\ \bar{F}_{23}\left(x_{1,k}^{(1)},+h,s_m,+h\right) & \bar{F}_{23}\left(x_{1,k}^{(1)},+h,s_m,-H\right) \\ \bar{F}_{21}\left(x_{1,k}^{(2)},-H,s_m,+h\right) & \bar{F}_{21}\left(x_{1,k}^{(2)},-H,s_m,-H\right) \\ \bar{F}_{22}\left(x_{1,k}^{(2)},-H,s_m,+h\right) & \bar{F}_{22}\left(x_{1,k}^{(2)},-H,s_m,-H\right) \\ \bar{F}_{23}\left(x_{1,k}^{(2)},-H,s_m,+h\right) & \bar{F}_{23}\left(x_{1,k}^{(2)},-H,s_m,-H\right) \end{bmatrix}, \tag{38}$$

where

$$\bar{W}_m = W_m\frac{1+\bar{s}_m^2}{\left(1-\bar{s}_m^2\right)^2}, \quad x_{1,k} = \frac{\bar{t}_k}{1-\bar{t}_k^2}, \quad s_m = \frac{\bar{s}_m}{1-\bar{s}_m^2}. \tag{39}$$

In this case, $\bar{b}_{+h}(\bar{s}_m,+h)$ and $\bar{b}_{-H}(\bar{s}_m,-H)$ are bounded at both ends of the integral, From Hills et al. (1996), the integration points $\bar{s}_m$, the collocation points $\bar{t}_k$ and the weight coefficients W_m can be calculated as

$$\bar{s}_m = \cos\frac{\pi m}{N+1}, \tag{40}$$

$$\bar{t}_k = \cos\frac{\pi(2k-1)}{2N+1}, \tag{41}$$

$$W_m = (1-\bar{s}_m^2)/(N+1). \tag{42}$$

where $k = 1,2,3,\ldots,(N+1)$; $m = 1,2,3,\ldots,N$ and N is the number of integration points. More detailed derivation of the numerical quadrature schemes for the solution of singular integral equations can be found in Hills et al. (1996). From Eq. (29), we can calculate $\mathbf{F}_{ij}(x_{1,k}^{(1)},+h,\bar{s}_m,+h)$, $\mathbf{F}_{ij}(x_{1,k}^{(1)},+h,\bar{s}_m,-H)$, $\mathbf{F}_{ij}(x_{1,k}^{(2)},-H,\bar{s}_m,+h)$ and $\mathbf{F}_{ij}(x_{1,k}^{(2)},-H,\bar{s}_m,-H)$.

From Eq. (37), we can obtain the dislocation arrays densities $\begin{bmatrix}\bar{\mathbf{b}}_{+h}(\bar{s}_m,+h)\\ \bar{\mathbf{b}}_{-H}(\bar{s}_m,-H)\end{bmatrix}$, which are related to the single dislocation $\mathbf{b}_0(x_{10},x_{20})$.

For convenience, we normalize the results. Denote the dislocation densities along $x_2=+h$ as $\bar{\mathbf{b}}_{+h1}$ due to $\mathbf{b}_{01}=\{1,0,0\}^{\mathrm{T}}$, $\bar{\mathbf{b}}_{+h2}$ due to $\mathbf{b}_{02}=\{0,1,0\}^{\mathrm{T}}$ and $\bar{\mathbf{b}}_{+h3}$ due to $\mathbf{b}_{03}=\{0,0,1\}^{\mathrm{T}}$ respectively. Similarly, we denote the dislocation densities along $x_2=-H$ as $\bar{\mathbf{b}}_{-H1}$ due to $\mathbf{b}_{01}=\{1,0,0\}^{\mathrm{T}}$, $\bar{\mathbf{b}}_{-H2}$ due to $\mathbf{b}_{02}=\{0,1,0\}^{\mathrm{T}}$ and $\bar{\mathbf{b}}_{-H3}$ due to $\mathbf{b}_{03}=\{0,0,1\}^{\mathrm{T}}$. Superposing the two elastic fields in Fig. 2, we can obtain the stress field for a single dislocation $\mathbf{b}_0=\{b_1,b_2,b_3\}^{\mathrm{T}}$ located at an arbitrary point $Z=x_{10}+\mathrm{i}x_{20}$ in the infinite strip as

$$\sigma_{ij}(x_1.x_2) = \bar{\mathbf{F}}_{ij}(x_1,x_2,x_{10},x_{20})\mathbf{b}(x_{10},x_{20}), \tag{43}$$

where

$$\bar{\mathbf{F}}_{ij}(x_1,x_2,x_{10},x_{20}) = [\bar{f}_{1ij}(x_1,x_2,x_{10},x_{20}),\bar{f}_{2ij}(x_1,x_2,x_{10},x_{20}),\bar{f}_{3ij}(x_1,x_2,x_{10},x_{20})], \tag{44}$$

$$\bar{f}_{lij}(x_1,x_2,x_{10},x_{20}) = f_{lij}(x_1,x_2,x_{10},x_{20}) + [\bar{\mathbf{F}}_{lij}(x_1,x_2,s_m,+h),\bar{\mathbf{F}}_{lij}(x_1,x_2,s_m,-H)] \times\langle\bar{W}_m\rangle\begin{bmatrix}\bar{\mathbf{b}}_{+hl}(\bar{s}_m,+h)\\ \bar{\mathbf{b}}_{-Hl}(\bar{s}_m,-H)\end{bmatrix}. \tag{45}$$

The first part of Eq. (45) is the stress components due to a single dislocation $\mathbf{b}_0 = \{b_1, b_2, b_3\}^{\mathrm{T}}$ located at an arbitrary point $z = x_{10} + \mathrm{i}x_{20}$ in the infinite plane; the second part is the stress components due to the two free boundaries of the infinite strip. The stress components of the infinite strip due to a single dislocation can be obtained by superposing the two parts together.

2.3. Mixed-mode stress intensity factors for interface cracks and cracks parallel to the interface in a bi-material infinite strip

A crack of length $2a$ in an infinite strip is shown in Fig. 3. The crack is parallel to the interface at a distance y_t. We denote T_{21}, T_{22} and T_{23} the external load distributing along the crack surface location. Cracks can be modelled as a dislocation array with the dislocation densities $\mathbf{b}(s, y_t)$.

The tractions along the crack surfaces due to the dislocation array are:

$$\sigma_{ij}^{c}(x_1, y_t) = \int_{-a}^{+a} \bar{\mathbf{F}}_{ij}(x_1, y_t, s, y_t)\mathbf{b}(s, y_t)\,\mathrm{d}s, \quad ij = 21, 22, 23, \tag{46}$$

which should be equal and opposite to the external loads T_{21}, T_{22} and T_{23}, in order to satisfy the traction free condition on the crack surfaces. We use the Gaussian formula to solve Eq. (46); then the singular integral equation can be transformed to $3(N-1)$ linear algebraic equations as

$$\pi a \bar{\mathbf{F}}_{ij}(x_{1,k}, y_t, s_m, y_t)\langle W_m \rangle \bar{\mathbf{b}}_{\mathrm{crack}}(\bar{s}_m, y_t) = -T_{ij}(x_{1,k}, y_t), \quad ij = 21, 22, 23, \tag{47}$$

where $s_m = a\bar{s}_m$, $x_{1,k} = a\bar{t}_k$.

Because the dislocation densities at the limits of the integration Eq. (46) are singular, that is to say there exists singularity at the ends of the crack, the integration points $\bar{s}_m$, the collocation points $\bar{t}_k$ and the weight coefficients W_m are (Huang and Kardomateas, 2001):

$$\bar{s}_m = \cos[\pi(2m-1)/2N], \quad m = 1, 2, 3, \ldots, N, \tag{48}$$

$$\bar{t}_k = \cos(\pi k/N), \quad k = 1, 2, 3, \ldots, N-1, \tag{49}$$

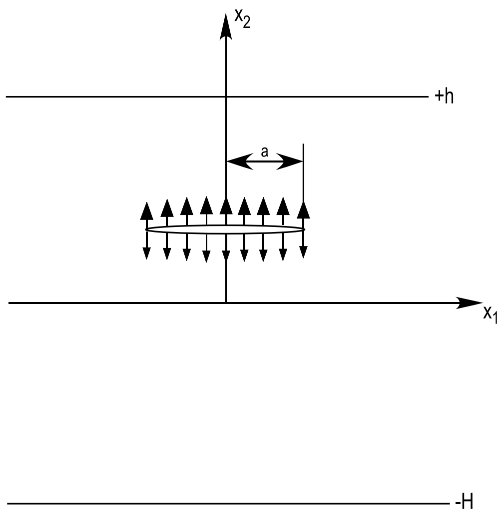

Fig. 3. A crack in an infinite strip.

$$W_m = 1/N, \tag{50}$$

where N is the number of the integration points we choose on the surfaces of the crack.

In order to satisfy the conditions that the crack surfaces physically come together at both ends, there are three equations:

$$\sum_{m=1}^{N} W_m \bar{b}_l(\bar{s}_m, y_t) = 0, \quad l = 1, 2, 3. \tag{51}$$

From the Eqs. (47)–(51), we can calculate the dislocation densities $\bar{b}(\bar{s}_m, y_t)$ at these N integration points along the crack surfaces. The crack tip dislocation densities can be extrapolated from these N integration points as

$$\bar{b}_l(1, y_t) = M_E \sum_{m=1}^{N} b_E^{(+1)} \bar{b}_l(\bar{s}_m, y_t), \tag{52}$$

$$\bar{b}_l(-1, y_t) = M_E \sum_{m=1}^{N} b_E^{(-1)} \bar{b}_l(\bar{s}_{N+1-m}, y_t), \tag{53}$$

where

$$b_E^{(+1)} = \sin\left[\frac{2m-1}{4N}\pi(2N-1)\right] \Big/ \sin\left(\frac{2m-1}{4N}\pi\right), \tag{54}$$

$$b_E^{(-1)} = b_E^{(+1)}; \quad M_E = \frac{1}{N}. \tag{55}$$

$l = 1, 2, 3$ (Hills et al., 1996).

The stress intensity factors at the crack tip can be calculated as (Huang and Kardomateas, 2001):

$$K(\pm 1, y_t) = [K_{\mathrm{II}}, K_{\mathrm{I}}, K_{\mathrm{III}}]^{\mathrm{T}} = \pm \frac{\sqrt{\pi a}}{2} Re\left\{ L_{i\alpha}\left[M_{\alpha j} d_j(\pm 1, y_t) + \delta(y_t) \sum_{j=1}^{3} E_{\alpha j}(\pm 1, y_t) \right] \right\}, \tag{56}$$

where $Re[\;]$ stands for the real part of a complex variable and $\delta(y_t)$ is the Dirac delta function. $E_{\alpha j}$ and d_j are solved from Eqs. (25), (26) and (15).

3. Results and discussion

First, we can validate the results by assuming a homogeneous material and selecting $h \gg H$, which would be essentially a homogeneous anisotropic half plane, since the analytical solution for a dislocation in a homogenous anisotropic half plane is presented by Atkinson and Eftaxiopoulos (1991). Cross-ply composite materials are studied in the paper. The elastic material properties for graphite/epoxy were taken from Huang and Kardomateas (2001). We list the material properties in Table 1. The fiber orientation is defined as the angle between the x_1 direction and the laminate's longitudinal direction.

Table 1
Material properties for graphite/epoxy laminate

$E_L = 134.45$ GPa, $E_T = 11.03$ GPa, $E_N = 11.03$ GPa
$G_{LT} = 5.84$ GPa, $G_{LN} = 5.84$ GPa, $G_{TN} = 2.98$ GPa
$\mu_{LT} = 0.301$, $\mu_{LN} = 0.301$, $\mu_{TN} = 0.49$

L is the longitudinal direction (fiber direction), T the transverse direction, and N the normal direction.

The convergence of the numerical integration method is very important, therefore, this is our first check. The results are listed in Table 2. A 45° homogeneous laminate is used in the analysis. In order to check the results with the analytical solutions obtained by Atkinson and Eftaxiopoulos (1991), we assume one of the boundaries located at $x_2 = 1.0 \times 10^9$; and the other free boundary located at $x_2 = -1.0$. The dislocation solutions of the infinite strip can be compared with a half plane because $h \gg H$. We assume a single dislocation $\mathbf{b} = \{1, 0, 0\}^T$ located at $z_0 = 0$. We check the stress components at the arbitrary points $z = 1$, 5, 10. From Table 2, we can see that the convergence of this method is very satisfactory. The results agree very well with the analytical solutions.

In order to further check the validity of this method, we assume a bi-material, but we take both h and H very large, therefore approaching an infinite bi-material plane. The analytical solutions for a single dislocation in a bi-material anisotropic infinite plane are known (Atkinson and Eftaxiopoulos's, 1991). Therefore, in Table 3, we show the comparison of the present solutions with corresponding analytical results. We assume two free boundaries located at $x_2 = 1.0 \times 10^9$, and $x_2 = -1.0 \times 10^9$ respectively. The dislocation solutions from this limiting case of the infinite strip compare well with the analytical solutions of the infinite plane. The material is chosen to be 0°/90°. We assume a single dislocation $\mathbf{b} = \{0, 1, 0\}^T$ located at $z_0 = 1$. We check the stress components of eight points around z_0. The number of integration points N is 300. The present results agree also very well with the analytical solutions.

Now we analyze next the mixed-mode stress intensity factors for an infinite strip. We assume the free boundaries located at $x_2 = +5$ and $x_2 = -5$ respectively. In order to simplify the results, we normalize the

Table 2
Convergence of stresses for a dislocation in homogeneous infinite strip material: 45°/45°, $H = 1$, $h = 1.0 \times 10^9$, dislocation $\mathbf{b} = \{1, 0, 0\}^T$ located at $z_0 = 0$

Stresses	$z = x_1 + ix_2$	Number of integration points N						Atkinson and Eftaxiopoulos (1991)
		10	50	100	150	200	250	
σ_{21}	1	1.6251	1.6333	1.6333	1.633	1.6333	1.6333	1.6333
	5	0.2732	0.2471	0.2547	0.255	0.255	0.255	0.255
	10	0.1009	0.0489	0.0613	0.0644	0.0652	0.0654	0.0655
σ_{22}	1	−0.1536	−0.1536	−0.1634	−0.1634	−0.1634	−0.1634	−0.1634
	5	0.0214	0.0552	0.0524	0.0521	0.0521	0.0521	0.0521
	10	−0.0293	0.0197	0.0142	0.0109	0.0096	0.0091	0.0089
σ_{23}	1	−0.959	−0.9607	−0.9607	−0.9607	−0.9607	−0.9607	−0.9607
	5	−0.1854	−0.1706	−0.1717	−0.1717	−0.1717	−0.1717	−0.1717
	10	−0.0619	−0.0427	−0.0467	−0.0473	−0.0474	−0.0474	−0.0474

Table 3
Comparison between present method and analytical solution of Atkinson and Eftaxiopoulos (1991) material: 0°/90°, $h = H = 1.0 \times 10^9$, dislocation $\mathbf{b} = \{0, 1, 0\}^T$ located at $z_0 = 1$

$z = x_1 + ix_2$	Present			Analytical solution		
	σ_{21}	σ_{22}	σ_{23}	σ_{21}	σ_{22}	σ_{23}
0.0 − 0.5i	−0.1516	−1.0913	0.0000	−0.1516	−1.0913	0.0000
0.5 − 0.5i	0.0985	−1.9325	0.0000	0.0985	−1.9325	0.0000
1.0 − 0.5i	2.0310	0.0000	0.0000	2.0310	0.0000	0.0000
1.5 − 0.5i	0.0985	1.9325	0.0000	0.0985	1.9325	0.0000
0.0 + 0.5i	0.1005	−1.0658	0.0000	0.1005	−1.0658	0.0000
0.5 + 0.5i	−0.3467	−1.6843	0.0000	−0.3467	−1.6843	0.0000
1.0 + 0.5i	−2.0310	0.0000	0.0000	−2.0310	0.0000	0.0000
1.5 + 0.5i	−0.3467	1.6843	0.0000	−0.3467	1.6843	0.0000

external loads as $\{T_{21}, T_{22}, T_{23}\}^{\mathrm{T}} = \{1, 1, 1\}^{\mathrm{T}}$. We check the stress intensity factors for the right crack tip. Fig. 4a shows the mode-I stress intensity factors for the interfacial crack in the infinite strip with the length of the crack from 0.5 to 5. The stress intensity factors are normalized as $\bar{K} = K/(\sigma\sqrt{\pi a})$, where σ is the external tensile load.

From the Fig. 4a, we can see that the material combination affects the mode-I stress intensity factor. The homogenous 0° material has the lowest mode-I stress intensity factor and the 90° material has the highest; the mode-I stress intensity factors are very close to each other for the homogeneous 45° material and the 45°/−45° bi-material.

The mode mixities ψ are defined as

$$\psi_{\mathrm{II}} = \tan^{-1}(K_{\mathrm{II}}/K_{\mathrm{I}}), \quad \psi_{\mathrm{III}} = \tan^{-1}(K_{\mathrm{III}}/K_{\mathrm{I}})$$

In Fig. 4b and c, we show the mode mixities ψ_{II} and ψ_{III}, respectively, as a function of the length of the crack. The mode mixities decrease as the crack length increases. Regarding the mode-II mixity, in general

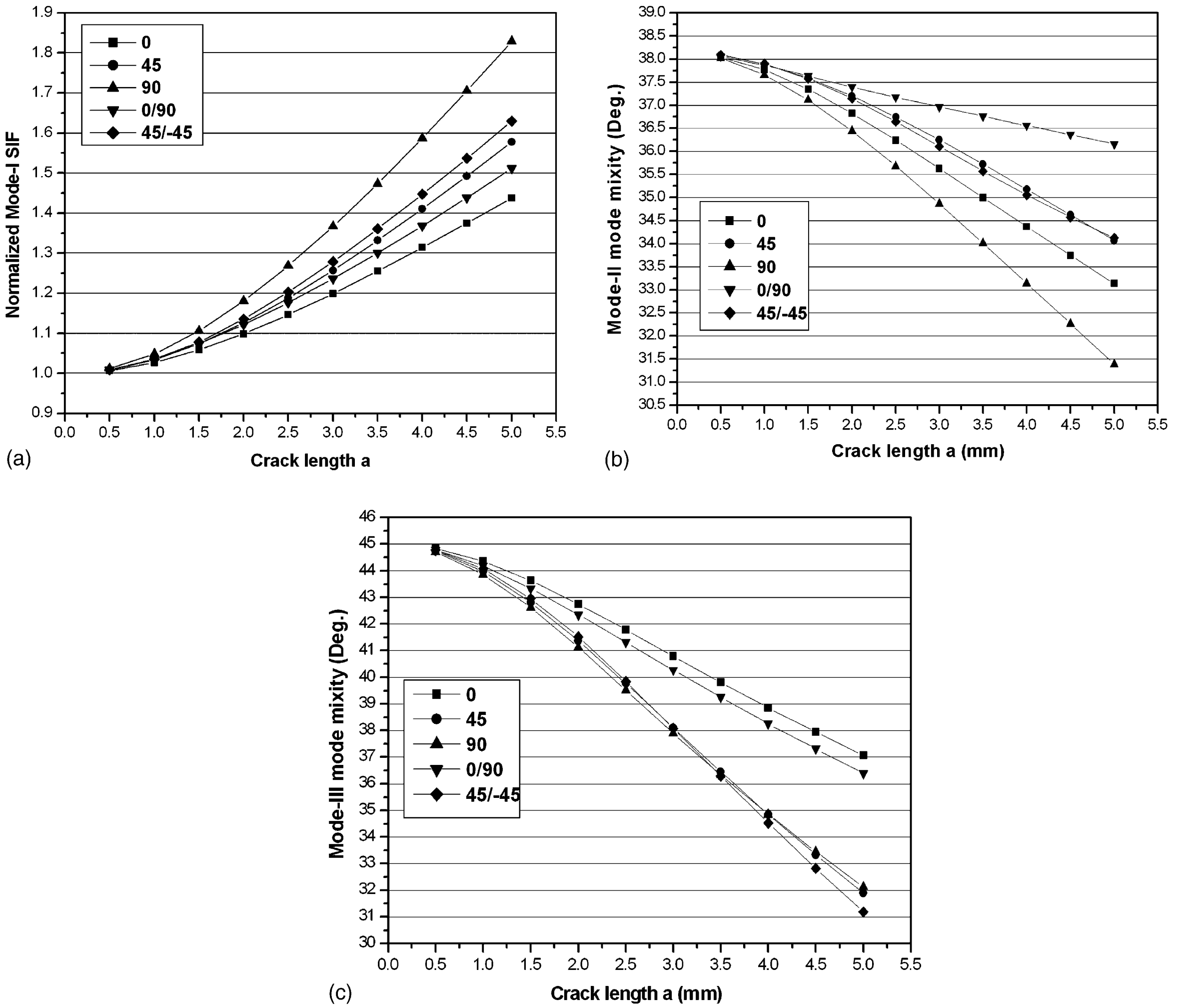

Fig. 4. (a) Normalized mode-I stress intensity factor for a crack located at the interface of a bi-material strip. (b) Mode-II mixity for a crack located at the interface of a bi-material strip. (c) Mode-III mixity for a crack located at the interface of a bi-material strip.

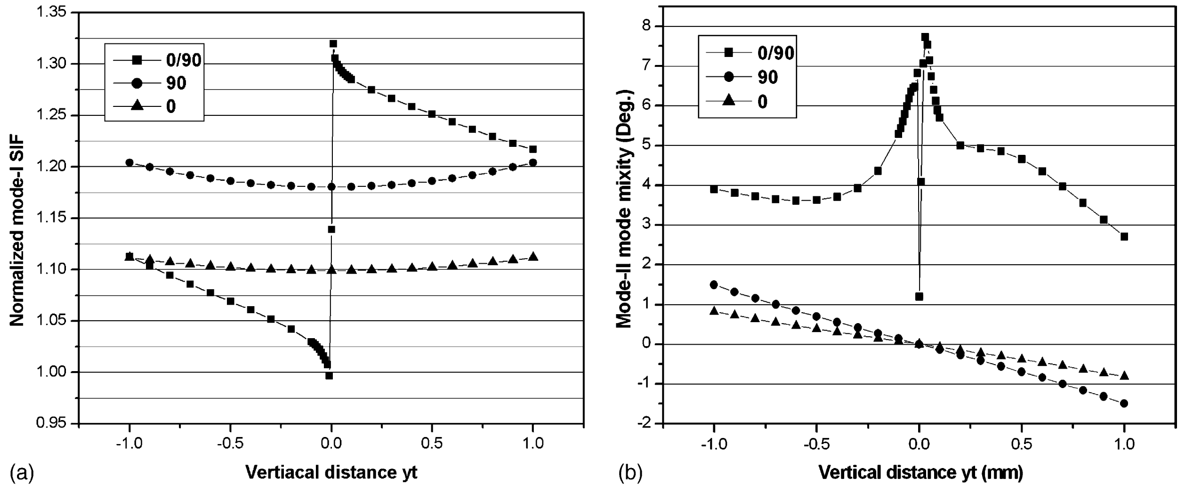

Fig. 5. (a) Normalized mode-I stress intensity factor for a crack parallel to the interface in a bi-material strip. (b) Mode-II mixity for a crack parallel to the interface in a bi-material strip.

the homogeneous materials has lower values compared with bi-material. Moreover, the homogeneous 90° material has the lowest mode-II mixity compared with the other homogeneous materials (0° and 45°. Regarding the mode-III mixity, it also decreases as the crack length increases; the 0° homogeneous case shows the highest mode-III mixity and the 45°/−45° bi-material has in general the lowest mode-III mixity compared with the other material combinations.

Fig. 5a,b shows the stress intensity factors for cracks parallel to the material interface. The vertical distance between the crack tip and the interface is y_t, shown in Fig. 3. The crack length is 2 and the vertical distance is from 1.0 to −1.0. The normalized external load is $\{T_{21}, T_{22}, T_{23}\}^T = \{1, 1, 1\}^T$. We compare three anisotropic materials: the 0° and 90° homogeneous materials and the $0^0/90^0$ bi-material. From the figures, we can see that the normalized mode-I stress intensity factor and mode-II mode mixities vary smoothly for the homogenous materials; however for the bi-material, they change drastically near the interface. When the crack is far away from the interface, the mode I SIF of the bi-material approaches that of the homogenous materials. The mode II mode mixities of the homogeneous materials are smaller than that of the bi-materials. The mode mixity of the interfacial crack is smaller than that of cracks located near the interface for the bi-material. These trends are similar to the ones observed by Huang and Kardomateas (2001) for the bi-material half plane.

Finally, it should be mentioned that the dislocation method presented in this paper can be extended to solve crack problems in bi-material finite-sized geometries, provided the anisotropic material is elastic and superposition is valid. In this case, the dislocation-based boundary element method (BEM) can be used, as outlined for the case of a homogeneous body by Huang and Kardomateas (2003).

4. Conclusions

Solutions for the stress intensity factors of cracks in a bi-material anisotropic infinite strip are derived based on the analytical dislocation approach. The accuracy and convergence are verified by considering the limits of a half plane and an infinite plane (i.e. for a very large thickness of the strip) and for the homogeneous case (i.e. the two materials to be identical), for which solutions already exist. We use the method to calculate the stress fields and the stress intensity factors for interfacial cracks and for cracks parallel to the interface. The following specific conclusions are drawn:

(1) The material combination affects the mode-I stress intensity factor and the mode mixities. The homogenous 0° material has the lowest mode-I stress intensity factors and the 90° material has the highest. As far as the mode mixities, the homogeneous 90° material has the lowest mode-II mixity and the 45°/−45° bi-material has the lowest mode-III mixity.
(2) For the cracks parallel to the interface, the mode-I and II SIFs change abruptly through the interface; the mode-II mixity of an interfacial crack is smaller than cracks near the interface.

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